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LETTER TO THE EDITOR

Invariants of particle motion in one-dimensional time-dependent potentials

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Abstract. We consider in this work one-dimensional time-dependent Hamiltonians of the form $H = p^2/2 + V(q, t)$. It is shown that the potentials determined in an implicit way by the relation $\Phi(V) = (a_1 V^2 + a_2 V + a_3)t + q$, where Φ is an arbitrary function and a_1, a_2 and a_3 are arbitrary parameters, admit a single-valued constant of motion. This constant of motion is a higher transcendental function in the momentum.

In the last few years, considerable attention has been devoted in search of constants of motion for time-dependent Hamiltonian systems (see, for example, references [1-26]). The search has been motivated both by an interest in understanding the structure of dynamical systems and by the possibility of applying exact invariants in subjects like plasma physics or quantum theory.

We consider in this work, one-dimensional Hamiltonians of the form

$$H = \tfrac{1}{2}p^2 + V(q, t) \tag{1}$$

where the potential $V(q, t)$ may depend explicitly both on the coordinate q and the time t. By the term invariant (or constant of motion) we mean any function that is constant along phase-space trajectories of the motion. That is, a function $I(q, p, t)$ is an invariant for a Hamiltonian $H(q, p, t)$ if it satisfies the condition:

$$\frac{\partial I}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial I}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial I}{\partial p} = 0. \tag{2}$$

Invariants of Hamiltonians of the form (1) are often very important in plasma physics when the description of the plasma system can be reduced to an equivalent one-dimensional problem. The equation (2) for the invariant is precisely the Liouville equation (known in plasma physics as the collisionless Boltzmann equation or the Vlasov equation). In the plasma context, $I(q, p, t)$ has the interpretation of a particle distribution function in the phase space [27].

The invariants of the classical Hamiltonian (1) can also be used to obtain exact solutions of time-dependent quantum mechanical problems. The invariant I then becomes an invariant quantum mechanical operator with eigenvalues that are constant in time. The first such solution was given by Lewis and Riensenfeld for the time-dependent harmonic oscillator [28]. Moreover, the Feynman propagator of the quantum problem can be written in terms of the eigenfunctions of the invariant I, as has been shown in [29].

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In general, solutions of the equations of motion associated with Hamiltonian (1) present chaotic behaviour. In contrast, for the special class of potentials that admit a single-valued constant of motion, the solutions of the corresponding dynamical equations will be regular and amenable to an analytical description, especially with respect to their asymptotic long-time behaviour.

From this point of view, the characterization of potentials that admit a single-valued invariant is equivalent to determining the Hamiltonians that present regular (non-chaotic) trajectories in phase space.

Methods that have been used for deriving invariants of Hamiltonian systems, as given in (1), include Noether's theorem, the Lie theory of extended groups, the theory of canonical transformations and the direct method. The direct method consists simply in making an ansatz about the form of the functional dependence of the invariant $I(q, p, t)$ on its arguments and then solving the defining equation (2) for an invariant. This method has been the more efficient and the simpler one for finding potentials that admit a constant of motion. By applying this procedure, potentials that admit invariants which are polynomial or rational functions in the momentum, have been found [18, 22, 24].

In this letter, by applying the direct method, we find a family of potentials that admit constants of motion which are higher transcendental functions in the momentum.

Taking into account (1) and the canonical equations of motion

$$\dot{q} = p \qquad \text{and} \qquad \dot{p} = -\frac{\partial V}{\partial q}$$

equation (2) for the invariant can be written as

$$\frac{\partial I}{\partial t} + p\frac{\partial I}{\partial q} - \frac{\partial V}{\partial q}\frac{\partial I}{\partial p} = 0. \tag{3}$$

The problem posed in this work is the following: which are the potentials that admit an invariant of the form $I(V, p)$, i.e. a function only of V and p? For such type of constants of motion, equation (3) becomes

$$\frac{\partial V}{\partial t}\frac{\partial I}{\partial V} + p\frac{\partial V}{\partial q}\frac{\partial I}{\partial V} - \frac{\partial V}{\partial q}\frac{\partial I}{\partial p} = 0. \tag{4}$$

It is evident that, in order to ensure the coherence of this equation, the following condition must be imposed on the potential:

$$\frac{\partial V}{\partial t} = f(V)\frac{\partial V}{\partial q} \tag{5}$$

where f is an arbitrary function.

By applying the method of characteristics for first-order partial differential equations [30], we find the general solution of (5):

$$\Phi(V) = f(V)t + q \tag{6}$$

where the function Φ can be chosen arbitrarily.

Therefore, expression (6) defines, in an implicit way, the class of potentials that admit an invariant which is a function only of V and p. When equation (5) is substituted in (4), the following equation for I is obtained:

$$(f(V) + p)\frac{\partial I}{\partial V} - \frac{\partial I}{\partial p} = 0. \tag{7}$$

The method of characteristics transforms the problem of solving (7) into the problem of solving an ordinary first-order differential equation, given by:

$$\frac{dV}{dp}+f(V)+p=0. \tag{8}$$

This equation is not integrable for an arbitrary function f. The more general function that enables us to solve (8) is

$$f(V)=a_1V^2+a_2V+a_3. \tag{9}$$

For this case, equation (8) becomes:

$$\frac{dV}{dp}+a_1V^2+a_2V+p+a_3=0 \tag{10}$$

which is a particular case of the Riccati equation. By means of the well known change of dependent variable

$$V(p)=\frac{1}{a_1u}\frac{du(p)}{dp} \tag{11}$$

(10) is transformed into a linear second-order differential equation given by:

$$\frac{d^2u}{dp^2}+a_2\frac{du}{dp}+a_1(p+a_3)u=0. \tag{12}$$

In order to eliminate the term that contains the first derivative in (12), we make a new change of dependent variable defined by:

$$u(p)=\exp\left(-\frac{a_2}{2}p\right)v(p) \tag{13}$$

after which we obtain

$$\frac{d^2v}{dp^2}+\left(a_1p+a_1a_3-\frac{a_2^2}{4}\right)v=0. \tag{14}$$

Finally, by introducing a new independent variable as follows

$$x=a_1^{-2/3}\left(\frac{a_2^2}{4}-a_1a_3-a_1p\right) \tag{15}$$

(14) is transformed into the Airy equation [31]

$$\frac{d^2y}{dx^2}-xy=0 \tag{16}$$

where

$$y(x)=v\left(\frac{a_2^2}{4a_1}-a_3-a_1^{-1/3}x\right) \tag{17}$$

and the bracket contains the argument of the function v.

The general solution of (16) is:

$$y(x)=\alpha_1\,\mathrm{Ai}(x)+\alpha_2\,\mathrm{Bi}(x) \tag{18}$$

where α_1 and α_2 are arbitrary constants and $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are two independent particular solutions of (16), called the Airy functions [31]. Taking into account expressions (11), (13), (17) and (18), we find the general solution of equation (10), which is given by

$$V(p)=\frac{-\frac{1}{2}a_2(K\,\mathrm{Ai}(x)+\mathrm{Bi}(x))-a_1^{1/3}(K\,\mathrm{Ai}'(x)+\mathrm{Bi}'(x))}{a_1(K\,\mathrm{Ai}(x)+\mathrm{Bi}(x))} \tag{19}$$

where x is given by (15), the primes indicate a derivative with respect to x, and $K=\alpha_1/\alpha_2$ is the arbitrary constant of integration of (10). Following the procedure indicated by the method of characteristics, the parameter K must be expressed in terms of the other quantities that appear in (19):

$$K=-\frac{(a_1V+\frac{1}{2}a_2)\,\mathrm{Bi}(x)+a_1^{1/3}\,\mathrm{Bi}'(x)}{(a_1V+\frac{1}{2}a_2)\,\mathrm{Ai}(x)+a_1^{1/3}\,\mathrm{Ai}'(x)}. \tag{20}$$

Finally, using (20), we can calculate the general solution of (7), with f given by (9)

$$I(q,p,t)=\psi\left(\frac{(a_1V+\frac{1}{2}a_2)\,\mathrm{Bi}(x)+a_1^{1/3}\,\mathrm{Bi}'(x)}{(a_1V+\frac{1}{2}a_2)\,\mathrm{Ai}(x)+a_1^{1/3}\,\mathrm{Ai}'(x)}\right) \tag{21}$$

where ψ is an arbitrary function. But a function of a constant of motion is also a constant of motion, and therefore we can write finally

$$I(q,p,t)=\frac{(a_1V(q,t)+\frac{1}{2}a_2)\,\mathrm{Bi}(x)+a_1^{1/3}\,\mathrm{Bi}'(x)}{(a_1V(q,t)+\frac{1}{2}a_2)\,\mathrm{Ai}(x)+a_1^{1/3}\,\mathrm{Ai}'(x)}. \tag{22}$$

It is interesting to analyse the particular case $a_1=0$. This case cannot be obtained from expression (22) by taking $a_1=0$, as a consequence of the resulting singularity in equation (15), which defines the quantity x. This particular case can be studied directly from equation (10), which for $a_1=0$ becomes

$$\frac{\mathrm{d}V}{\mathrm{d}p}+a_2V+p+a_3=0 \tag{23}$$

i.e. a linear first-order equation. Its general solution is

$$V=K\,\mathrm{e}^{-a_2p}-\frac{p}{a_2}+\frac{1}{a_2^2}-\frac{a_3}{a_2} \tag{24}$$

from which we find the constant of motion

$$I(q,p,t)=\mathrm{e}^{a_2p}\left(V(q,t)+\frac{p}{a_2}+\frac{a_3}{a_2}-\frac{1}{a_2^2}\right) \tag{25}$$

which presents a very simple dependence in the momentum p, when compared to the general case (22).

In summary, in this work we have found a family of time-dependent one-dimensional potentials, determined in an implicit way by relations (8) and (9), which admit a constant of motion given by (22). For each function Φ and parameters a_1, a_2 and a_3, a potential V of this family is determined. The corresponding constant of motion is, in the general case, a higher transcendental function in the momentum. Invariants that are quadratic functions in the momentum have been used to find exact solutions of the Vlasov–Poisson equations in plasma physics [27]. It would be interesting to analyse if the invariants that have been found in this work, especially for the case $a_1=0$, where the p-dependence of the invariant is more simple, could be used to find new solutions of the Vlasov–Poisson equations.

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